Zadanie 4. Znajdź (bez wspomagania komputerowego) najmniejsze $n > 0$ takie, że $3^n - 2^n$ jest wielokrotnością liczby 175. *Uwaga:* Mile widziane będzie zapisanie rozwiązania tego zadania w języku angielskim.

Solution. Given that $175 = 25 \cdot 7$ (and $25 \perp 7$), ChRT gives us equvalent system of congruences.

$$\begin{cases} 3^n \equiv 2^n \mod 7; \\ 3^n \equiv 2^n \mod 25. \end{cases}$$

The first congruence $(\mathsf{mod}\ 7)$ is satisfied iff $6 \mid n$. To show this, we begin by writing down the sequence of $(3^n \mathsf{\ mod\ } 7)$ and $(2^n \mathsf{\ mod\ } 7)$.

n	3^n mod 7	2^n mod 7
1	3	2
2	2	4
3	6	1
4	4	2
5	5	4
6	**1**	**1**
7	3	2

Note that the sequence of residues is cyclic and $3^n \equiv 2^n \pmod 7$ holds iff the residues on both sides of congruence are equal, that is, iff $6 \mid n$.

For the second congruence, we do the same.

n	3^n mod 25	2^n mod 25
1	3	2
2	9	4
3	2	8
4	6	16
5	18	7
6	4	14
7	12	3
8	11	6
9	8	12
10	$-\mathbf{1}$	$-\mathbf{1}$
11	-3	-2

Thus $3^n \equiv 2^n \pmod{25}$ iff $10 \mid n$. So, to satisfy the system of congruences (and accordingly the equivalent big congruence), the n must satisfy both $6 \mid n$ and $10 \mid n$ at the same time, that is, it has to be the (lowest) common multiple of these two numbers.

$$n = \mathsf{LCM}(10, 6) = \frac{10 \cdot 6}{\mathsf{GCD}(10, 6)} = 30$$